

## **On the Law of Entropy Increase of Some Cellular Automata on $Z^d$**

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We consider some time-reversible cellular automata on the  $d$ -dimensional integral lattice  $Z^d$  and study their time evolution properties. We show first that a Boltzmann-type entropy can be defined which is not less than its initial value for initial states which have no spatial correlation. For monotonic increase of the entropies for such initial states we need an additional condition which we call renewality. Under the renewality condition entropy is monotonic non-decreasing. We give some examples of cellular automata which satisfy the renewality condition.

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**KEY WORDS:** Law of entropy increase; reversible cellular automaton.

### **1. INTRODUCTION**

Among the fundamental problems of statistical mechanics one of the utmost interest is the justification of the second law of thermodynamics, or the explanation of the apparent conflict between microscopic dynamical reversibility and macroscopic irreversibility.<sup>(3, 4, 6, 9, 10, 11)</sup>

For the isolated finite dynamical system it is impossible, as is well known, to define rigorously a nonequilibrium entropy as a dynamical variable if we require that a law of increasing entropy should be strictly realized. But these circumstances do not deny the possibility to be able to prove the increasing law of the entropy rigorously, if we consider the systems of infinitely many degrees of freedom and restrict the initial states. Note that it is very natural to consider the infinite systems, since the second law of thermodynamics itself has statistical nature, and so the size of the number of particles should be taken into account.<sup>(2, 4, 5, 6, 9, 11)</sup>

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In the papers<sup>(9, 10)</sup> we considered a one dimensional hard-points infinite system on  $Z$  whose particles have several colors and velocities with unit magnitude. We showed there the system has many interesting time evolution properties, especially we showed that Boltzmann type entropies increase monotonically for the initial states which have no spatial correlation.

In this paper we expand these results to more general systems. Namely, we consider some time reversible cellular automata on the  $d$ -dimensional integral lattice  $Z^d$  and study their time evolution properties. We show firstly that Boltzmann type entropies of the systems do not decrease for the initial nonequilibrium states which have no spatial correlation. This result which holds for rather general systems does not mean that the entropies increase monotonically. As is known,<sup>(4, 11)</sup> monotonical increase of the entropy and the reversibility of the system "contradict" each other. Therefore more important and difficult question is that under what type of interaction and for what type of initial (nonequilibrium) states their entropies of the reversible system increase monotonically.

We will show that one of the natural candidates of such interactions is the one which we call renewality.

It is known that if the system is mixing then the "Boltzmann" entropy can increase asymptotically for nonequilibrium states.<sup>(11)</sup> However our assertion of entropy increase is not merely asymptotic but monotonic. Our systems are time reversible, so monotonical increase of entropy for all the initial nonequilibrium states is impossible, even if the systems are of infinite degrees of freedom.<sup>(4, 11)</sup> Therefore it is necessary to restrict the initial states to have the monotonic increase of entropy. We believe our locally equilibrium states, namely states with no spatial correlations are such a natural class.

In Section 2, we describe the class of systems and the definitions of their entropies. In Section 3, we consider general reversible systems not necessary with renewality, state some results and prove them. In Section 4 and 5 we give our main results, Theorems 4 and 5, namely we define cellular automata of renewal type for which the entropies increase monotonically for the initial conditions without spatial correlation and construct concretely some cellular automata of renewal type.

## 2. DESCRIPTION OF DYNAMICS OF CELLULAR AUTOMATON AND THE DEFINITIONS OF ITS ENTROPIES

In this section we define a dynamics of cellular automaton and its entropies.

Let  $S$  and  $Y$  be finite sets, and  $\mathbf{Z}^d$  be the  $d$ -dimensional integral lattice. The *configuration space*  $\Omega$  of our *cellular automaton* (CA) is the product space of  $S$  over  $Y \times \mathbf{Z}^d$ :

$$\Omega := S^{Y \times \mathbf{Z}^d} = \prod_{(y,z) \in Y \times \mathbf{Z}^d} S_{y,z}, \quad S_{y,z} := S \quad \text{for } \forall (y,z) \in Y \times \mathbf{Z}^d \quad (1)$$

We call  $Y$  *momentum space* and its elements *momentum*, which represent “momentum” or “velocity” of the particles. Usually we take a subset of  $\mathbf{Z}^d$  as  $Y$ . We call  $S$  *state space* and its elements *state*, which represent the “state” or “colour” of the particles.  $S_{y,z}$  is the state space over momentum-lattice site  $(y,z) \in Y \times \mathbf{Z}^d$ .

A configuration  $\omega$ , an element of  $\Omega$  is written as follows:

$$\omega = \{ \omega(y,z) \}_{(y,z) \in Y \times \mathbf{Z}^d}, \quad \omega(y,z) \in S \quad (2)$$

$\omega(y,z)$  indicates the state or colour of the particle on the lattice site  $z \in \mathbf{Z}^d$  with momentum  $y \in Y$ .

For  $\omega \in \Omega$  we denote its restriction on a subset  $R \subset \mathbf{Z}^d$  by  $\omega_R$ , that is,

$$\omega_R(y,z) = \omega(y,z) \quad \forall (y,z) \in Y \times R \subset Y \times \mathbf{Z}^d$$

The time evolution of our CA is given by a transition function, which defines the dynamical rule of the CA. The *transition function*  $\Phi$  is a function from  $Y \times S^{Y \times R_0}$  to  $S$ , where  $R_0$  is a finite set of  $\mathbf{Z}^d$ :

$$\Phi: Y \times S^{Y \times R_0} \rightarrow S: (y, \omega_{R_0}) \mapsto \Phi(y, \omega_{R_0}) \quad (3)$$

where  $\omega_{R_0} \in S^{Y \times R_0}$ . We call  $R_0$  the *range of interaction*.

The *time evolution map*  $T \equiv T_\Phi$  on  $\Omega$  is defined as follows: for  $\forall \omega = \{ \omega(y,z) \} \in \Omega$

$$(T\omega)(y,z) = \Phi(y, \omega_{R_0+z}) \quad (4)$$

here  $R_0 + z = \{ y + z; y \in R_0 \}$ .

The map  $T$  is called *reversible* when it is a bijection of  $\Omega$ , that is, a one to one mapping of  $\Omega$  onto itself, and the inverse mapping  $T^{-1}$  is also given by some transition function,  $\Phi^{-1}: Y \times S^{Y \times R_0} \rightarrow S$ , with the same range of interaction.

The *macrostate space*  $\mathcal{M}$  of the CA,  $(\Omega, \Phi)$  or  $(\Omega, T)$  consists of all Borel probability measures  $\mu$  on  $\Omega$ .  $T$  induces a map of  $\mathcal{M}$ , which we denote also by  $T$ , defined by

$$(T\mu)(A) := \mu(T^{-1}A) \quad (\mu \in \mathcal{M}, A \subset \Omega) \quad (5)$$

We denote the distribution of the densities on  $S_{y,z}$  with respect to  $\mu \in \mathcal{M}$ , by  $\rho_{y,z}(\mu)$  or  $\rho_{y,z}(\cdot; \mu)$ , that is,

$$\rho_{y,z}(s; \mu) := \mu\{\omega; \omega(y, z) = s\} \quad (s \in S_{y,z}). \tag{6}$$

Let  $\bar{\mathcal{M}}$  be the subset of  $\mathcal{M}$  which consists of probabilities with no spatial correlations, that is, direct product probabilities on each momentum-lattice site:

$$\bar{\mathcal{M}} := \left\{ \mu \in \mathcal{M}; \mu = \prod_{(y,z) \in Y \times \mathbf{Z}^d} \mu_{y,z} \right\} \tag{7}$$

where  $\mu_{y,z}$  is a probability measure on  $S_{y,z} ((y, z) \in Y \times \mathbf{Z}^d)$ .

An element of  $\bar{\mathcal{M}}$  will be called a *locally equilibrium macrostate* on  $\Omega$ , and  $\bar{\mathcal{M}}$  will be called a *locally equilibrium macrostate space*.

We can define the natural *projection*  $\pi$  from  $\mathcal{M}$  to  $\bar{\mathcal{M}}$  as follows:

For  $\mu \in \mathcal{M}$ ,  $\pi(\mu) \in \bar{\mathcal{M}}$  is a direct product probability measure  $\pi(\mu) = \prod_{(y,z) \in Y \times \mathbf{Z}^d} \mu_{y,z}$ , where  $\mu_{y,z}$  is the probability on  $S_{y,z}$  defined by

$$\mu_{y,z}(s) := \rho_{y,z}(s; \mu) \quad \text{for } \forall s \in S$$

Note that

$$\rho_{y,z}(\cdot; \pi\mu) = \rho_{y,z}(\cdot; \mu) \quad \text{for } \forall \mu \in \mathcal{M}$$

Now let us define the *Boltzmann-type entropy*  $H(\mu)$  of  $\mu \in \mathcal{M}$  by

$$H(\mu) := \limsup_{N \rightarrow \infty} \frac{-1}{|A_N|} \sum_{(y,z) \in A_N} \sum_{s \in S} \rho_{y,z}(s; \mu) \log \rho_{y,z}(s; \mu) \tag{8}$$

where  $A_N := \{(y, z) = (y, z_1, z_2, \dots, z_d) \in Y \times \mathbf{Z}^d; |z_i| \leq N, i = 1, 2, \dots, d\}$ ,  $|A_N| := (2N + 1)^d |Y|$ ,  $|Y|$  is the number of the elements of  $Y$ .<sup>(2)</sup>

Secondly we define an auxiliary *KS (Kolmogorov-Sinai)-type entropy*  $h(\mu)$  of  $\mu \in \mathcal{M}$  by

$$h(\mu) := \limsup_{N \rightarrow \infty} \frac{-1}{|A_N|} \sum_{A \in \mathcal{C}(A_N)} \mu(A) \log \mu(A) \tag{9}$$

Here  $\mathcal{C}(A_N)$  is the partition of  $\Omega$  defined by the states on  $A_N$ :<sup>(1)</sup>

$$\mathcal{C}(A_N) := \{A = \omega(A_N) \times S^{Y \times \mathbf{Z}^d - A_N}; \omega(A_N) \in S^{A_N}\}$$

Note that our *KS-type entropy*  $h(\mu)$  is different from the entropy which appears in the second law of thermodynamics. We believe our

Boltzmann type entropy  $H(\mu)$  relates directly to it. It may be better to call  $h(\mu)$  the “information” of the macrostate  $\mu$  which represents the complexity of the structure of  $\mu$ .

Note also that the measure  $\mu$  does not need to be translationally invariant. The entropies  $H(\mu), h(\mu)$  are defined by supremum limit, so they exist always for every  $\mu \in \mathcal{M}$ . Note that

$$H(\mu), h(\mu) \leq \log |S|$$

for  $|S|$ , the number of the elements of the set  $S$ .

### 3. LAW OF ENTROPY INCREASE

Now in general let  $\mu_k$  be a probability measure on a finite set  $S_k$ , ( $k = 1, \dots, K$ ). Let  $\mathcal{M}(\mu_1, \dots, \mu_K)$  be the set of probability measures  $\bar{\mu}$  on the product set  $S_1 \times \dots \times S_K$  such that  $\bar{\mu}\pi_k^{-1} = \mu_k$ , where  $\pi_k$  is the natural projection from the product set  $S_1 \times \dots \times S_K$  onto  $S_k$  ( $k = 1, \dots, K$ ). Note that the direct product probability of  $\mu_1, \dots, \mu_K$ ,  $\mu_1 \times \dots \times \mu_K \in \mathcal{M}(\mu_1, \dots, \mu_K)$ .

In general we define the entropy  $\varepsilon(\bar{\mu})$  of a probability measure  $\bar{\mu}$  on a finite set  $\bar{S}$  in the usual way by

$$\varepsilon(\bar{\mu}) := - \sum_{\bar{s} \in \bar{S}} \bar{\mu}(\bar{s}) \log \bar{\mu}(\bar{s}) \tag{10}$$

The following Lemma 1 (“the *maximal entropy principle*”) is well known and can be easily proven from the definition of the entropy  $\varepsilon(\bar{\mu})$  by using Jensen’s inequality.

**Lemma 1.** For  $\forall \bar{\mu} \in \mathcal{M}(\mu_1, \dots, \mu_K)$ , we have

$$\varepsilon(\bar{\mu}) \leq \varepsilon(\mu_1 \times \dots \times \mu_K) = \varepsilon(\mu_1) + \dots + \varepsilon(\mu_K) \tag{11}$$

The equality holds iff  $\bar{\mu} = \mu_1 \times \dots \times \mu_K$ .

From the inequality of the Lemma 1 we obtain easily the following

**Proposition 1.** For  $\forall \mu \in \mathcal{M}$ , we have

$$h(\pi\mu) \geq h(\mu) \tag{12}$$

The equality holds if  $\mu \in \bar{\mathcal{M}}$ .

As a corollary of the equality of the Lemma 1 we obtain following

**Proposition 2.** For  $\forall \mu \in \mathcal{M}$ , we have

$$H(\mu) = h(\pi\mu) \tag{13}$$

*Proof.* Let  $\mu_{y,z}$  be the probability measure on  $S_{y,z}$  defined by  $\mu_{y,z}(s) = \rho_{y,z}(s; \pi\mu)$ . Then by the equality of the Lemma 1,

$$\begin{aligned} & - \sum_{A \in \mathcal{G}(A_N)} \pi\mu(A) \log \pi\mu(A) \\ &= \varepsilon \left( \prod_{(y,z) \in A_N} \mu_{y,z} \right) \\ &= \sum_{(y,z) \in A_N} \varepsilon(\mu_{y,z}) \\ &= - \sum_{(y,z) \in A_N} \left( \sum_{s \in S_{y,z}} \rho_{y,z}(s; \pi\mu) \log \rho_{y,z}(s; \pi\mu) \right) \\ &= - \sum_{(y,z) \in A_N} \left( \sum_{s \in S_{y,z}} \rho_{y,z}(s; \mu) \log \rho_{y,z}(s; \mu) \right) \end{aligned}$$

Hence we have  $h(\pi\mu) = H(\mu)$ . ■

Now let us consider our dynamical system  $(\Omega, T)$ .

**Theorem 1.** KS-type entropy  $h(\mu)$  does not increase for  $\forall \mu \in \mathcal{M}$ :

$$h(T\mu) \leq h(\mu) \tag{14}$$

In addition, if  $T$  be reversible, then the KS-type entropy  $h(\mu)$  does not change in time for  $\forall \mu \in \mathcal{M}$ :

$$h(T\mu) = h(\mu) \tag{15}$$

*Proof.* As the range of the interaction  $R_0$  is a finite set, so there exists  $N_0$  such that  $Y \times R_0 \subset A_{N_0}$ . From the assumption it is easy to see that for  $\forall N > N_0$  the states of  $T\omega$  on  $(y, z) \in A_N$ ,  $T\omega(y, z)$  are determined by the states  $\omega(y', z')$  on  $(y', z') \in A_{N+N_0}$ . Conversely if  $T$  is reversible the states  $\omega$  on  $(y, z) \in A_{N-N_0}$  are determined from the states  $T\omega(y', z')$  on  $(y', z') \in A_N$ . Hence

$$\mathcal{C}(A_{N+N_0}) \supset T^{-1}\mathcal{C}(A_N) \supset \mathcal{C}(A_{N-N_0})$$

(When  $T$  is not reversible we get only the first inequality.)

Here  $\mathcal{A} > \mathcal{B}$  ( $\mathcal{A}, \mathcal{B}$  are partitions of  $\Omega$ ) means that  $\mathcal{A}$  is a refinement of  $\mathcal{B}$ , that is, for  $\forall A \in \mathcal{A}$  there exists a  $B \in \mathcal{B}$  such that  $A \subset B$ .

It is well known that if  $\mathcal{A} > \mathcal{B}$  then

$$-\sum_{A \in \mathcal{A}} \mu(A) \log \mu(A) \geq -\sum_{B \in \mathcal{B}} \mu(B) \log \mu(B)$$

Therefore

$$\begin{aligned} & -\sum_{A \in \mathcal{G}(A_{N+N_0})} \mu(A) \log \mu(A) \\ & \geq -\sum_{B \in T^{-1}\mathcal{G}(A_N)} \mu(B) \log \mu(B) \geq -\sum_{C \in \mathcal{G}(A_{N-N_0})} \mu(C) \log \mu(C) \end{aligned}$$

Hence from the first inequality we have

$$\begin{aligned} h(\mu) &= \limsup_{N \rightarrow \infty} \frac{-1}{|A_{N+N_0}|} \sum_{A \in \mathcal{G}(A_{N+N_0})} \mu(A) \log \mu(A) \\ &\geq \limsup_{N \rightarrow \infty} \frac{-1}{|A_N|} \sum_{B' \in \mathcal{G}(A_N)} \mu(T^{-1}B') \log \mu(T^{-1}B') \\ &= \limsup_{N \rightarrow \infty} \frac{-1}{|A_N|} \left( \sum_{B' \in \mathcal{G}(A_N)} (T\mu)(B') \log (T\mu)(B') \right) \frac{|A_N|}{|A_{N+N_0}|} \\ &= h(T\mu) \end{aligned}$$

Note that  $\lim_{N \rightarrow \infty} |A_N|/|A_{N+N_0}| = 1$ .

Similarly we get from the second inequality

$$h(T\mu) \geq h(\mu) \quad \blacksquare$$

Theorem 1 tells us that when  $T$  is reversible  $T$  does not lose the “information” of the state  $\mu \in \mathcal{M}$ . But when  $T$  is not reversible, *i.e.*, not one to one, then it may lose the “information”. As for the  $H(\mu)$  we have following

**Theorem 2** (“Weak law of entropy increase”).

Assume that  $T$  is reversible, then the Boltzmann type entropy of any locally equilibrium macrostate  $\mu \in \bar{\mathcal{M}}$  does not decrease from the initial one in the sense that:

$$H(T^n \mu) \geq H(\mu) \quad \text{for } \forall n \in \mathbf{Z} \tag{16}$$

*Proof.* From Proposition 1, 2 and Theorem 1

$$H(T^n\mu) \geq h(T^n\mu) = h(\mu)$$

As  $\mu \in \bar{\mathcal{M}}$  we have  $\pi\mu = \mu$ , so that

$$h(\mu) = h(\pi\mu) = H(\mu)$$

This proves the theorem. ■

Theorem 2 only asserts the non-decrease of the entropy  $H(T^n\mu)$  at time  $n \in \mathbf{Z}$  from the initial entropy  $H(\mu)$  for the locally equilibrium macrostate  $\mu \in \bar{\mathcal{M}}$ . We cannot assert the same result for general  $\mu \notin \bar{\mathcal{M}}$ . In fact, if we take as initial macrostate  $\mu = T^{-n_0}\mu_0$  for some  $\mu_0 \in \bar{\mathcal{M}}$  and  $n_0 > 0$ , then the entropy of  $\mu$  can even decrease:  $H(T^{n_0}\mu) \leq H(\mu)$ . [We can assert strict inequality holds. (See ref. [9])]

In order to get the monotonic increase of the entropy  $H(T^n\mu)$  with respect to the time  $n \geq 0$ , we have to restrict our class of dynamical rules or that of the transition functions  $\Phi$  which define the dynamical rules.

#### 4. DYNAMICAL RULES OF RENEWAL TYPE

Let  $D(S)$  be the set of all probability distributions on the state space  $S$ :

$$D(S) = \left\{ p: S \rightarrow \mathbf{R}; \sum_{s \in S} p(s) = 1, p(s) \geq 0 \text{ for } \forall s \in S \right\}. \quad (17)$$

**Definition.**  $\mu \in \mathcal{M}$  is called of *renewal type* with respect to the transition function  $\Phi$ , if the density distribution  $\rho_{y,z}(\cdot; T^{n+1}\mu)$  on each momentum-lattice site  $(y, z) \in Y \times \mathbf{Z}^d$  with respect to  $T^{n+1}\mu$  is determined only by the densities  $\{\rho_{y',z+r}(\cdot; T^n\mu)\}$  on the momentum-lattice sites  $(y', z+r) \in Y \times (z + R_0)$  with respect to  $T^n\mu$  for all  $n \geq 0$ , that is, if there exists a function

$$\begin{aligned} \phi: Y \times (D(S)^{Y \times R_0}) &\rightarrow D(S): (y, \{p_{y',r}\}_{(y',r) \in Y \times R_0}) \\ &\mapsto \phi(\cdot, y, \{p_{y',r}\}_{(y',r) \in Y \times R_0}) \end{aligned}$$

such that

$$\begin{aligned} \rho_{y,z}(s; T^{n+1}\mu) &= \phi(s, y, \{p_{y',z+r}(\cdot; T^n\mu)\}_{(y',r) \in Y \times R_0}) \\ \forall s \in S, \forall (y, z) \in Y \times \mathbf{Z}^d, \quad \text{and} \quad \forall n \geq 0 \end{aligned} \quad (18)$$



A transition function  $\Phi$  of CA or CA itself is called of *renewal type*, if all  $\mu \in \bar{\mathcal{M}}$  are of renewal type with respect to the transition function  $\Phi$ . We call the function  $\phi \equiv \phi^\Phi$  the *renewal function* of the transition function  $\Phi$ , and the above equation (18) the *renewal equation*.

Henceforth we will often omit the subscript  $(y', r) \in Y \times R_0$  for simplicity.

**Remark.** When a transition function  $\Phi$  is of renewal type, then its renewal function  $\phi$  is determined uniquely from the transition function  $\Phi$  itself, namely

$$\phi(\cdot, y, \{p_{y', r}\}) = \rho_{y, o}(\cdot; T\mu)$$

where  $\mu \in \bar{\mathcal{M}}$  is such a locally equilibrium state that

$$\rho_{y', r}(\cdot; \mu) = p_{y', r} \quad \text{for } \forall (y', r) \in Y \times R_0$$

and  $o = (0, \dots, 0) \in Z^d$ .

Note that for  $\forall \{p_{y', r} \in D(S)\}$ , there exists such a  $\mu \in \bar{\mathcal{M}}$  that

$$\rho_{y', r}(\cdot; \mu) = p_{y', r} \quad \text{for } \forall (y', r) \in Y \times R_0$$

and  $\rho_{y, o}(\cdot; T\mu)$  is uniquely determined by  $\{p_{y', r}; (y', r) \in Y \times R_0\}$ . In this sense we denote the renewal function of the transition function  $\Phi$  of renewal type by  $\phi = \phi^\Phi$ .

A trivial example of a transition function  $\Phi$  of renewal type is “translation,” that is, when the map  $T$  defined by  $\Phi$  is given by for some  $z_0 \in Z^d$

$$(T\omega)(y, z) = \omega(y, z + z_0) \quad \text{for } \forall (y, z) \in Y \times Z^d \quad (19)$$

We give some nontrivial examples of transition functions of renewal type in the next section.

**Theorem 3.** Assume  $T$  be defined by a transition function  $\Phi$  of renewal type, then we have for any locally equilibrium macrostate  $\mu \in \bar{\mathcal{M}}$

$$\pi(T(\pi(T^n\mu))) = \pi(T^{n+1}\mu) \quad \text{for } \forall n \geq 0 \quad (20)$$

*Proof.* It is sufficient to prove that for  $\forall (y, z) \in Y \times Z^d$ ,

$$\rho_{y, z}(\cdot; T^{n+1}\mu) = \rho_{y, z}(\cdot; T(\pi T^n\mu))$$

From the assumption we have

$$\rho_{y,z}(s; T^{n+1}\mu) = \phi^\Phi(s, y, \{\rho_{y',z+r}(\cdot; T^n\mu)\}), \quad \forall s \in \mathcal{S},$$

where  $\phi^\Phi$  is the renewal function of the  $\Phi$ .

As  $\pi T^n\mu \in \bar{\mathcal{M}}$  we have from the above remark

$$\begin{aligned} \rho_{y,z}(s; T(\pi T^n\mu)) &= \phi^\Phi(s, y, \{\rho_{y',z+r}(\cdot; \pi T^n\mu)\}) \\ &= \phi^\Phi(s, y, \{\rho_{y',z+r}(\cdot; T^n\mu)\}) \end{aligned}$$

Hence the conclusion follows. ■

As a corollary of the Theorem 3, we have following

**Theorem 4** (“Law of entropy increase”). Assume  $T$  to be reversible and defined by a transition function  $\Phi$  of renewal type, then the Boltzmann type entropy of any locally equilibrium macrostate  $\mu \in \bar{\mathcal{M}}$  increases monotonically in the sense that:

$$H(T^{n+1}\mu) \geq H(T^n\mu) \quad \text{for } \forall n \geq 0 \quad (21)$$

*Proof.* From Proposition 2, we get

$$H(T^{n+1}\mu) = h(\pi T^{n+1}\mu)$$

As  $\mu \in \bar{\mathcal{M}}$  so by Theorem 3, we have

$$h(\pi T^{n+1}\mu) = h(\pi T \pi T^n\mu)$$

Then by Proposition 1

$$h(\pi T \pi T^n\mu) \geq h(T \pi T^n\mu)$$

Finally by Theorem 1 and Proposition 2, we have

$$h(T \pi T^n\mu) = h(\pi T^n\mu) = H(T^n\mu)$$

This proves the theorem. ■

**Important Remark:** Theorem 4 claims only non-decrease of entropy for reversible  $T$  of renewal type, but we can construct a CA of

renewal type for which Boltzmann type entropies for some  $\mu \in \bar{\mathcal{M}}$  actually increase strictly:<sup>(9)</sup>

$$H(T^{n+1}\mu) > H(T^n\mu) \quad \text{for } \forall n \geq 0$$

(See the remark at the end of Theorem 5.)

### 5. EXAMPLES OF CELLULAR AUTOMATA OF RENEWAL TYPE

Firstly we give a simple example of a transition function of renewal type. We will call the CA with such a transition function “ideal gas.”

The state space  $S$  and the momentum space  $Y$  are arbitrary finite sets. Let  $\{r_y \in \mathbf{Z}^d; y \in Y\}$  be a function from  $Y$  to  $\mathbf{Z}^d$ . The transition function  $\Phi^{i.g.}$  of our ideal gas is given by

$$\Phi^{i.g.}(y, \{\omega(y', r)\}) := \omega(y, r_y)$$

The range of interaction  $R_0$  is  $R_0 := \{r_y \in \mathbf{Z}^d; y \in Y\}$ .

In other words time evolution map  $T$  of our ideal gas is given by

$$(T\omega)(y, z) = \omega(y, z + r_y) \quad \text{for } \forall (y, z) \in Y \times \mathbf{Z}^d \quad (22)$$

A physical interpretation could be that  $\omega(y, z) = s$  means the particle on the site  $z \in \mathbf{Z}^d$  with “velocity”  $r_y$  is of colour  $s$ . Each particle of the “ideal gas” is translated independently.

**Proposition 3.** The transition function of ideal gas  $\Phi^{i.g.}$  is of renewal type, and its renewal function  $\phi^{i.g.}$  is given by

$$\phi^{i.g.}(\cdot, y, \{p_{y', r}\}) := p_{y, r_y} \quad (23)$$

*Proof.* From the definition of the time evolution map,

$$(T^n\omega)(y, z) = \omega(y, z + nr_y)$$

Therefore we get

$$\rho_{y, z}(\cdot; T^n\mu) = \rho_{y, z + nr_y}(\cdot; \mu)$$

Hence for  $\forall n \geq 0$ , we have

$$\rho_{y, z}(\cdot; T^{n+1}\mu) = \rho_{y, z + r_y}(\cdot; T^n\mu)$$

This means the renewal equation

$$\rho_{y,z}(\cdot; T^{n+1}\mu) = \phi^{i.g.}(\cdot, y, \{\rho_{y',z+r}(\cdot; T^n\mu)\})$$

holds with renewal function,

$$\phi^{i.g.}(\cdot, y, \{p_{y',r}\}) := p_{y,r}$$

Hence the conclusion follows. ■

Note that in the ideal gas case, the entropy  $H(T^n\mu)$  is time invariant for all  $\mu \in \bar{\mathcal{M}}$ :

$$H(T^n\mu) = H(\mu) \quad \text{for } \forall n \in \mathbf{Z}$$

To have strict increase,  $H(T^{n+1}\mu) > H(T^n\mu)$ , we need to construct more complicated interaction. So we give another example of a transition function of renewal type which has an interaction of a ‘‘collision’’ type and gives strict increase of entropy. This is a slightly generalized system of the one which was considered in refs. [9, 10].

Let  $Y := \{-1, +1\} \subset \mathbf{Z}$ ,  $S := \{0, 1, 2, \dots, k\}$  ( $k \geq 2$ ), and  $\mathbf{Z}^d = \mathbf{Z}$  ( $d = 1$ ).

The transition function  $\Phi$  is defined by

$$\Phi(y, \omega_{R_0}) := \begin{cases} \sigma_y \omega(-y, y) & \text{if } \omega(y, -y) \cdot \omega(-y, y) \neq 0 \\ \omega(y, -y) & \text{if } \omega(y, -y) \cdot \omega(-y, y) = 0 \end{cases}$$

where  $\sigma_y$  ( $y \in Y$ ) are permutations of  $S$  such that  $\sigma_y(0) = 0$ . The range of interaction is  $R_0 := \{-1, +1\} \subset \mathbf{Z}$ . We denote this transition function  $\Phi$  by  $\Phi = \Phi^{col}$ .

In other words, the time evolution map  $T$  of our CA is given by

$$T\omega(y, z) = \begin{cases} \sigma_y \omega(-y, z+y) & \text{if } \omega(y, z-y) \cdot \omega(-y, z+y) \neq 0 \\ \omega(y, z-y) & \text{if } \omega(y, z-y) \cdot \omega(-y, z+y) = 0 \end{cases} \quad (24)$$

Intuitively saying,  $\omega(y, z) = s$  ( $s \neq 0$ ) means there exists a particle of the colour  $s$  on the site  $z \in \mathbf{Z}$  with velocity  $y = \pm 1$ , while  $\omega(y, z) = 0$  means there exists no such particle. Particles behave like hard-points when  $\sigma_y = \text{identity}$  for all  $y \in Y$ .  $\{\sigma_y\}$  denotes a kind of chemical reaction, which makes the change of colours of the particles after collision.

**Theorem 5.** The above defined transition function  $\Phi^{col}$  is of renewal type, and its renewal function  $\phi^{col}$  is given by

$$\begin{aligned} &\phi^{col}(s, y, \{p_{y', r}\}) \\ &:= \begin{cases} p_{y, -y}(0) & \text{if } s = 0 \\ p_{-y, y}(0) p_{y, -y}(s) + (1 - p_{y, -y}(0)) p_{-y, y}(\sigma_y^{-1} s) & \text{if } s \neq 0 \end{cases} \end{aligned}$$

**Lemma 2.** For the above defined transition function  $\Phi^{col}$ ,

$$\begin{aligned} \{T^{n+1}\omega(y, z) = 0\} &= \{T^n\omega(y, z - y) = 0\} = \{\omega(y, z - (n + 1)y) = 0\} \\ &\forall n \geq 0 \quad \text{and} \quad \forall (y, z) \in Y \times Z \end{aligned}$$

and

$$\begin{aligned} \{T^{n+1}\omega(y, z) = s\} &= (\{T^n\omega(-y, z + y) = \sigma_y^{-1} s\} \\ &\quad \cap \{\omega(y, z - (n + 1)y) \neq 0\}) \\ &\quad \cup (\{T^n\omega(y, z - y) = s\} \\ &\quad \cap \{\omega(-y, z + (n + 1)y) = 0\}) \\ &\quad \forall s \neq 0, \quad \forall n \geq 0 \quad \text{and} \quad \forall (y, z) \in Y \times Z \end{aligned}$$

where union  $\cup$  is the disjoint sum.

*Proof.* The first part of the lemma is a direct consequence of the definition of the transition function  $\Phi^{col}$ .

Now let  $s \neq 0$ , we divide the set of configurations  $\{T^{n+1}\omega(y, z) = s\}$  according to whether the state  $T^{n+1}\omega(-y, z)$  on the momentum-lattice site  $(-y, z)$  is 0 or not, namely,

$$\begin{aligned} &\{T^{n+1}\omega(y, z) = s\} \\ &= (\{T^{n+1}\omega(y, z) = s\} \cap \{T^{n+1}\omega(-y, z) \neq 0\}) \\ &\quad \cup (\{T^{n+1}\omega(y, z) = s\} \cap \{T^{n+1}\omega(-y, z) = 0\}) \end{aligned}$$

By the definition of the transition function  $\Phi^{col}$ , we have

$$\begin{aligned} &\{T^{n+1}\omega(y, z) = s\} \cap \{T^{n+1}\omega(-y, z) \neq 0\} \\ &= \{T^n\omega(-y, z + y) = \sigma_y^{-1} s\} \cap \{T^n\omega(y, z - y) \neq 0\} \end{aligned}$$

From the first part of the lemma

$$\{T^n\omega(y, z - y) \neq 0\} = \{\omega(y, z - (n + 1)y) \neq 0\}$$

hence

$$\begin{aligned} & \{T^{n+1}\omega(y, z) = s\} \cap \{T^{n+1}\omega(-y, z) \neq 0\} \\ &= \{T^n\omega(-y, z+y) = \sigma_y^{-1}s\} \cap \{\omega(y, z-(n+1)y) \neq 0\} \end{aligned}$$

Similarly we have

$$\begin{aligned} & \{T^{n+1}\omega(y, z) = s\} \cap \{T^{n+1}\omega(-y, z) = 0\} \\ &= \{T^n\omega(y, z-y) = s\} \cap \{T^n\omega(-y, z+y) = 0\} \\ &= \{T^n\omega(y, z-y) = s\} \cap \{\omega(-y, z+(n+1)y) = 0\} \end{aligned}$$

These prove the lemma. ■

*Proof of Theorem 5.* Let  $s \neq 0$ . From Lemma 2,

$$\begin{aligned} \rho_{y,z}(s; T^{n+1}\mu) &= \mu\{T^{n+1}\omega(y, z) = s\} \\ &= \mu(\{T^n\omega(-y, z+y) = \sigma_y^{-1}s\} \cap \{\omega(y, z-(n+1)y) \neq 0\}) \\ &\quad + \mu(\{T^n\omega(y, z-y) = s\} \cap \{\omega(-y, z+(n+1)y) = 0\}) \end{aligned}$$

By the definition of  $\Phi^{col}$ , we can see easily that the state  $T^n\omega(-y, z+y)$  on the momentum-lattice site  $(-y, z+y)$  at time  $n$  is determined by the states  $\{\omega(y', z')\}$  on the momentum-lattice sites  $\{(y', z'); y' \in Y, z' \in [z-(n-1)y, z+(n+1)y]\}$  at time 0. Here  $[a, b]$  denotes the set of such integers  $z$  that  $a \leq z \leq b$  (or  $b \leq z \leq a$ ). Hence the events  $\{T^n\omega(-y, z+y) = \sigma_y^{-1}s\}$  and  $\{\omega(y, z-(n+1)y) \neq 0\}$  are mutually independent with respect to the probability  $\mu \in \bar{\mathcal{M}}$ . Therefore we have

$$\begin{aligned} & \mu(\{T^n\omega(-y, z+y) = \sigma_y^{-1}s\} \cap \{\omega(y, z-(n+1)y) \neq 0\}) \\ &= \mu\{T^n\omega(-y, z+y) = \sigma_y^{-1}s\} \cdot \mu\{\omega(y, z-(n+1)y) \neq 0\} \\ &= \rho_{-y, z+y}(\sigma_y^{-1}s; T^n\mu)(1 - \rho_{y, z-(n+1)y}(0; \mu)) \end{aligned}$$

Similarly we have

$$\begin{aligned} & \mu(\{T^n\omega(y, z-y) = s\} \cap \{\omega(-y, z+(n+1)y) = 0\}) \\ &= \rho_{y, z-y}(s; T^n\mu) \rho_{-y, z+(n+1)y}(0; \mu) \end{aligned}$$

Hence we have for  $s \neq 0$

$$\begin{aligned} \rho_{y,z}(s; T^{n+1}\mu) &= \rho_{-y, z+(n+1)y}(0; \mu) \rho_{y, z-y}(s; T^n\mu) \\ &\quad + (1 - \rho_{y, z-(n+1)y}(0; \mu)) \rho_{-y, z+y}(\sigma_y^{-1}s; T^n\mu) \end{aligned}$$

Similarly from the first part of Lemma 2, we get

$$\rho_{y,z}(0; T^{n+1}\mu) = \rho_{y,z-y}(0; T^n\mu) = \rho_{y,z-(n+1)y}(0; \mu) \tag{25}$$

Now, by the definition of  $\phi = \phi^{col}$ , we have for  $s \neq 0$ ,

$$\begin{aligned} &\phi(s, y, \{\rho_{y',z+r}(\cdot; T^n\mu)\}) \\ &= \rho_{-y,z+y}(0; T^n\mu) \rho_{y,z-y}(s; T^n\mu) \\ &\quad + (1 - \rho_{y,z-y}(0; T^n\mu)) \rho_{-y,z+y}(\sigma_y^{-1}s; T^n\mu) \end{aligned}$$

which is equal to, from above equalities (25),

$$\begin{aligned} &= \rho_{-y,z+(n+1)y}(0; \mu) \rho_{y,z-y}(s; T^n\mu) \\ &\quad + (1 - \rho_{y,z-(n+1)y}(0; \mu)) \rho_{-y,z+y}(\sigma_y^{-1}s; T^n\mu) \end{aligned}$$

Therefore we get

$$\rho_{y,z}(s; T^{n+1}\mu) = \phi(s, y, \{\rho_{y',z+r}(\cdot; T^n\mu)\})$$

This proves the theorem for the case  $s \neq 0$ .

Now let  $s = 0$ . By the definition of  $\phi = \phi^{col}$ ,

$$\phi(0, y, \{\rho_{y',z+r}(\cdot; T^n\mu)\}) = \rho_{y,z-y}(0; T^n\mu)$$

which is, from (25), equal to

$$= \rho_{y,z}(0; T^{n+1}\mu)$$

This proves the theorem for  $s = 0$ . ■

**Important Remark:** We showed in ref. [9] that for CA defined by the transition function  $\Phi^{col}$  with trivial  $\sigma_y \equiv \text{identity}$  [ $\sigma_y(s) = s$  for  $\forall s \in S$ ], the Boltzmann type entropy  $H(\mu)$  actually increases strictly for “general”  $\mu \in \bar{\mathcal{M}}$ :

$$H(T^{n+1}\mu) > H(T^n\mu) \quad \text{for } \forall n \geq 0.$$

Since  $h(T^n\mu)$  is invariant,  $H(T^n\mu) - h(T^n\mu)$  increases monotonically for  $\mu \in \bar{\mathcal{M}}$ . Roughly speaking it represents the amount of mutual correlation between the  $\{\omega(y, z)\}$  on different sites  $z \in \mathbb{Z}^d$ . Note that the initial macrostate  $\mu \in \bar{\mathcal{M}}$  has no such correlation. The time evolution  $T^n\mu$  of  $\mu$  generates the correlations.

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